Explicit Local Class Field Theory à la Lubin and Tate with an Application to Algebraic Topology

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Contents

Quadratic Reciprocity Law

Suppose p, q are two distinct primes. Define the Legendre symbol by

$$\left(\frac{p}{q}\right) = \begin{cases} 1 & p \equiv x^2 \pmod{q} \text{ for some } x \in \mathbb{Z} \\ -1 & \text{otherwise} \end{cases}$$

Suppose further p, q are odd. Let $q^* := (-1)^{\frac{q-1}{2}} q$. The quadratic reciprocity law tells us

$$\left(\frac{p}{q}\right)\left(\frac{q^*}{p}\right) = 1$$

Quadratic Reciprocity Law in terms of Fields Extension

Consider the following surjective homomorphism, where $I(q)$ is the subgroup of $\mathbb{Q}^*$ generated by primes distinct to q:

$$\begin{aligned} \phi\colon I(q) &\to \{\pm 1\} = \mathsf{Gal}\big(\mathbb{Q}(\sqrt{q^*})/\mathbb{Q}\big) \\ p &\mapsto \left(\frac{q^*}{p}\right) \end{aligned}$$

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For any $p \in \mathrm{Nm}_{\mathbb{Q}(\sqrt{q^*})/\mathbb{Q}}\big(\mathbb{Q}(\sqrt{q^*})^*\big)$,

$$p = a^2 - b^2 q^*, a, b \in \mathbb{Q} \quad \Rightarrow \quad \left(\frac{p}{q}\right) = 1$$

By quadratic reciprocity, $\ker\phi \supset \mathrm{Nm}\big(\mathbb{Q}(\sqrt{q^*})^*\big)$.

One of the Main Theorems of Local Class Field Theory

Theorem (Local Reciprocity Law)

Suppose K is a non-archimedean local field. $\exists!\ \phi_K \colon K^* \to Gal(K^{ab}/K)$, s.t.

(a) $\forall \pi$ *uniformizer of K, $\phi_K(\pi)|_{K^{un}}$ is the Frobenius of $Gal(K^{un}/K)$.*

(b) *For any finite abelian extension L of K, there is an exact sequence:*

$$1 \to Nm_{L/K}(L^*) \to K^* \overset{\phi_K(\cdot)|_L}{\to} Gal(L/K) \to 1$$

$$\rightsquigarrow \ \phi_{L/K} \colon K^*/Nm_{L/K}(L^*) \ \overset{\sim}{\to} \ Gal(L/K)$$

*The map $\phi_{L/K}$ is then called the **local Artin map**.*

Lubin and Tate's Work

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In 1965, Lubin and Tate gave an explicit description of K^{ab} and ϕ_K via Lubin-Tate formal group law.

Motivation for the Construction

Given the isomorphisms

$$\phi_{L/K} \colon K^*/\mathsf{Nm}(L^*) \to \mathsf{Gal}(L/K) \cong \mathsf{Gal}(K^{ab}/K)/\mathsf{Gal}(K^{ab}/L)$$

for each finite abelian extension L of K.

Passing to the profinite completion, we get an isomorphism:

$$\hat{\phi}_K \colon \widehat{K^*} \to \mathsf{Gal}(K^{ab}/K)$$

On the other hand, we have a factorization: $\widehat{K^*} \cong \mathcal{O}_K^* \times \pi^{\hat{\mathbb{Z}}} \cong \mathcal{O}_K^* \times \hat{\mathbb{Z}}$.

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Let $K_\pi = (K^{ab})^{\hat{\phi}_K(\pi)}$ and $K^{un} = (K^{ab})^{\hat{\phi}_K(\mathcal{O}_K^*)}$.

By infinite Galois theory, $\mathsf{Gal}(K^{ab}/K_\pi) = \hat{\mathbb{Z}}$ and $\mathsf{Gal}(K^{ab}/K^{un}) = \mathcal{O}_K^*$.

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It can be shown that K_π/K is totally ramified,

and K^{un}/K is the maximal unramified extension in K^{ab}.

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$$\Rightarrow \qquad K^{un} \cap K_\pi = K$$

$$\text{and} \quad \mathrm{Gal}(K_\pi K^{un}/K) = \mathrm{Gal}(K_\pi/K) \times \mathrm{Gal}(K^{un}/K) = \mathcal{O}_K^* \times \hat{\mathbb{Z}}$$

Outline of the Proof

We know that $\phi_K(\pi)|_{K^{un}}$ is the Frobenius element.

The proof of local class field theory consists of several steps:

(a) Construct the fields K^{un}, K_π discussed above and the restriction of the local Artin map $\mathcal{O}_K^* \overset{\sim}{\to} \mathrm{Gal}(K_\pi/K)$.

(b) Extend the map to $\phi_\pi : K^* \to \mathrm{Gal}(K_\pi K^{un}/K)$.

(c) Show that ϕ_π is independent of the choice of π.

(d) Show that $K_\pi K^{un} = K^{ab}$.

(e) Show that ϕ_π satisfies the condition (b) of Local Reciprocity Law.

Contents

The Case of $\mathbb{Q}_p$

Suppose $K = \mathbb{Q}_p$ and pick the uniformizer $\pi = p$.

Local Kronecker-Weber Theorem:

$$\mathbb{Q}_p^{ab} = \bigcup_{n \in \mathbb{Z}_+} \mathbb{Q}_p(\mu_n) = \left(\bigcup_{(n,p)=1} \mathbb{Q}_p(\mu_n) \right) \cdot \left(\bigcup_{i=1}^{\infty} \mathbb{Q}_p(\mu_{p^i}) \right)$$

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Let

$$\mathbb{Q}_p^{un} = \bigcup_{(n,p)=1} \mathbb{Q}_p(\mu_n) \qquad (\mathbb{Q}_p)_\pi = \bigcup_{i=1}^{\infty} \mathbb{Q}_p(\mu_{p^i})$$

$\mathbb{Q}_p^{un}$ is the maximal unramified extension of $\mathbb{Q}_p$.

$(\mathbb{Q}_p)_\pi / \mathbb{Q}_p$ is totally ramified.

General Case

K^{un} can be obtained from K by adjoining n-th roots of unity ($p \nmid n$).

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For $(\mathbb{Q}_p)_\pi = \bigcup_{i=1}^{\infty} \mathbb{Q}_p(\mu_{p^i} - 1)$,

$\mu_{p^i} - 1$ can be viewed as a root of $(1+T)^{p^i} - 1$.

Let $F(X, Y) = X + Y + XY = (1+X)(1+Y) - 1$ be the multiplicative formal group law.

$(1+T)^{p^i} - 1 = [p^i]_F(T)$ is the p^i-series of F.

Then $(\mathbb{Q}_p)_\pi$ is obtained by adjoining p^i-th "division values" to $\mathbb{Q}_p$.

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Lubin and Tate generalized this idea to arbitrary K.

In particular, Lubin-Tate formal group laws generalize $F(X, Y)$.

Formal Group Laws

Definition (Commutative One-Parameter Formal Group Law)

Let R be a commutative ring. A **(commutative one-parameter) formal group law** is a power series $F \in R[\![X, Y]\!]$ satisfying that

(a) $F(X,Y) \equiv X + Y \pmod{(X,Y)^2}$.

(b) (Associativity) $F\big(X, F(Y,Z)\big) = F\big(F(X,Y), Z\big)$.

(c) (Commutativity) $F(X,Y) = F(Y,X)$.

Lubin-Tate Formal Group Laws

Definition

Let $\mathcal{F}_\pi$ be the set of $f(T) \in \mathcal{O}_K[\![T]\!]$ such that

(a) $f \equiv \pi T \pmod{T^2}$.

(b) $f \equiv T^q \pmod{\pi}$.

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Lemma (Lubin-Tate)

Suppose $f, g \in \mathcal{F}_\pi$ and $\phi_1(X_1, \cdots, X_n) \in \mathcal{O}_K[X_1, \cdots, X_n]$ is linear. Then there exists a unique $\phi \in \mathcal{O}_K[\![X_1, \cdots, X_n]\!]$ such that

(a) $\phi \equiv \phi_1 \pmod{(X_1, \cdots, X_n)^2}$.

(b) $f\big(\phi(X_1, \cdots, X_n)\big) = \phi\big(g(X_1), \cdots, g(X_n)\big)$.

Lubin-Tate Formal Group Laws

Corollary

For every $f \in \mathcal{F}_\pi$*, take* $\phi_1(X, Y) = X + Y$*.*

Then there is a unique formal group law $F_f \in \mathcal{O}_K[\![X, Y]\!]$ *such that* $f(F(X, Y)) = F(f(X, Y))$*, i.e.,* $f \in End(f)$

The formal group law F_f is called the **Lubin-Tate formal group law associated to** π **(and** f**)**.

Lubin-Tate Formal Group Laws

Corollary

For every $f \in \mathcal{F}_\pi$, take $\phi_1(X,Y) = X + Y$.

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Proposition

There is a ring isomorphism $\mathcal{O}_K \to$ End(F) given by $a \mapsto [a]_f(T)$ where $[a]_f(T) \equiv aT \pmod{T^2}$. In particular, $f = [\pi]_f$.

Lubin-Tate Formal Group Laws

Example

When $K = \mathbb{Q}_p$, $\pi = p$, $f(T) = (1+T)^p - 1$, $F_f = \mathbb{G}_m = X + Y + XY$ is the multiplicative formal group law. When $a \in \mathbb{Z}$, $[a]_f(T) = (1+T)^a - 1$. This can be extended to $\mathbb{Z}_p$. For any $a \in \mathbb{Z}_p$,

$$(1+T)^a := \sum_{m \geqslant 0} \binom{a}{m} T^m \qquad \binom{a}{m} := \frac{a(a-1)\cdots(a-m+1)}{m(m-1)\cdots 1}$$

Since $\mathbb{Z}_p$ is complete, $\binom{a}{m} \in \mathbb{Z}_p$ and $[a]_f(T) := (1+T)^a - 1 \in \mathrm{End}(\mathbb{G}_m)$.

Lubin-Tate Formal Group Laws

For any $f \in \mathcal{F}_\pi$, let $\Lambda_f = \{\alpha \in K^{al} : |\alpha| < 1\}$.

Define a $\mathcal{O}_K$-module structure on Λ_f by

$$\alpha + \beta := \alpha +_{F_f} \beta \text{ and } a \cdot \alpha := [a]_f(\alpha) \text{ for } a \in \mathcal{O}_K$$

Let $\Lambda_{f,n}$ be the submodule of Λ_f consisting of roots of $[\pi^n]_f$.

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Proposition

For each n, $\Lambda_{f,n} \cong \mathcal{O}_K/(\pi^n)$ as $\mathcal{O}_K$-modules.

Construction of K_π

Theorem

Let $K_{\pi,n} := K(\Lambda_{f,n})$. Then we have

(a) *$K_{\pi,n}$ is independent of the choice of f.*

(b) *For each n, $K_{\pi,n}/K$ is a totally ramified extension of degree $(q-1)q^{n-1}$.*

(c) *The action of $\mathcal{O}_K$ on $\Lambda_{f,n}$ induces an isomorphism*

$$(\mathcal{O}_K/\mathfrak{m}^n)^* \to \mathit{Gal}(K_{\pi,n}/K)$$

Let $K_\pi := \bigcup_{n\in\mathbb{Z}_+} K_{\pi,n}$.

Contents

Morava E-Theory

Definition (Deformation of a Formal Group Law and $\star$-Isomorphisms)

Let G be a formal group law over k and A is a complete local ring with the maximal ideal $\mathfrak{m}$ and residue field containing k. Suppose $\pi\colon A \to A/\mathfrak{m}$ is the natural projection and $i\colon k \to A/\mathfrak{m}$ is the inclusion. A **deformation of** G **to** A is a formal group law F over A, such that $\pi_*(F) = i_*(G)$.

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Let $F, \tilde{F}$ be two deformations of G over A. Then the two deformations are said to be **$\star$-isomorphic** if there is an isomorphism $\sigma\colon F \to \tilde{F}$ such that $\pi_*(\sigma) = T$. Then define

$$\mathrm{Def}(A, G) := \{F \text{ is a deformation of } G \text{ over } A\}/\star\text{-isomorphic}$$

Morava E-Theory

Theorem (Lubin-Tate)

For any formal group law G of height n over k, $\exists$ a universal formal group law F_{univ} over $\mathscr{R} := W(k)[\![v_1, \cdots, v_{n-1}]\!]$ such that for any complete local ring A with residue field containing k, there is a bijection

$$Hom_{/k}(\mathscr{R}, A) \to Def(A, F)$$
$$\phi \mapsto \phi_*(F_{univ})$$

Furthermore, $(v_0 = p, v_1, v_2, \cdots)$ is regular.

Morava E-Theory

Let $\Phi(T)$ be the Honda formal group law over $k = \mathbb{F}_p$ of height n, i.e., $[p]_\Phi(T) = T^q$, where $[p]_\Phi(T)$ is the p-series of Φ.

Morava E-Theory

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By Landweber exact functor theorem,

Definition (Morava E-Theory)

The generalized cohomology theory E_n with $(E_n)_* = \mathscr{R}[\beta^{\pm 1}]$, where $\deg(\beta) = -2$, classifying deformations of $\Phi(T)$ is called **Morava E-theory** E_n.

Power Operations

Fact: Both Morava E-theories and MU carry power operations, i.e., a multiplicative map $P_j^E : E^{2*}(X) \to E^{2*}(D_j X)$, where $D_j X := E\Sigma_j \times_{\Sigma_j} X^j$.

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A complex orientation on E_n is same to a map $t\colon \mathrm{MU} \to E_n$ between ring spectra.

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$$\begin{array}{ccc} \mathrm{MU}^{2*}(X) & \xrightarrow{P_j^{MU}} & \mathrm{MU}^{2j*}(D_j X) \\ \downarrow t & & \downarrow t \\ E_n^{2*}(X) & \xrightarrow[P_j^{E_n}]{} & E_n^{2j*}(D_j X) \end{array}$$

When does the diagram commute?

Ando's Theorem

Theorem (Ando)

In each $\star$-isomorphism class of lifting of Φ to the Lubin-Tate ring $\mathcal{R}$, there is a unique formal group law F satisfying

$$[p]_F(T) = \prod_{\lambda \text{ is a root of } [p]_F} (T +_F \lambda)$$

Moreover, the power operations on MU, E_n are compatible under t if and only if the formal group law F associated to t satisfies the condition.

Norm-Coherent Coordinates

Suppose $\mathscr{F}$ is the formal group associated to the universal formal group law F_{univ}. A coordinate on $\mathscr{F}$ is an isomorphism $\mathscr{F} \to \hat{\mathbb{A}}^1 = \mathrm{Spf}\mathscr{R}[\![T]\!]$. Recall a coordinate on $\mathscr{F}$ will give a formal group law over $\mathscr{R}$ via the multiplication map of $\mathscr{F}$.

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Suppose X is a coordinate on $\mathscr{F}$ such that the associated formal group law F lifts Φ.

$$[p]_F \colon \mathscr{F} \to \mathscr{F} \rightsquigarrow [p]_F^* \colon \mathcal{O}_{\mathscr{F}} \to \mathcal{O}_{\mathscr{F}}$$

Then F satisfies the condition if and only if

$$X = \mathsf{Nm}_{[p]_F^*}(X)$$

Proof of Ando's Theorem in a Special Case

Suppose K is a local field with uniformizer $\pi = p$,

i.e., K is an unramified extension of $\mathbb{Q}_p$.

For any $f \in \mathcal{F}_\pi$, F_f is a Lubin-Tate formal group law and $[p]_{F_f}(T) = f(T)$.

$\Rightarrow F_f$ is a lifting of Φ.

Conversely, every lifting of Φ to $\mathcal{O}_K$ has p-series in $\mathcal{F}_\pi$, so it is a

Lubin-Tate formal group law.

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Lubin-Tate formal group law.

$$\text{Liftings of } \Phi \text{ to } \mathcal{O}_K \iff \text{Lubin-Tate formal group law.}$$

Example

$K = \mathbb{Q}_p$ and $F(X, Y) = X + Y + XY$ is a Lubin-Tate formal group law.

$$[p]_F(T) = (1 + T)^p - 1 \equiv T^p \pmod{p}$$

Proof of Ando's Theorem in a Special Case

Theorem (Ando, a Special Case)

In each $\star$-isomorphism class of lifting of Φ to $\mathcal{O}_K$, there is a unique formal group law F_f satisfying

$$[p]_{F_f}(T) = \prod_{\lambda \in \Lambda_{f,1}} (T +_{F_f} \lambda)$$

Coleman Norm Operator

Theorem

There exists a unique $\mathcal{N}_{F_f}\colon \mathcal{O}_K((T)) \to \mathcal{O}_K((T))$ *satisfying*

$$\mathcal{N}_{F_f}(g) \circ [p]_{F_f} = \prod_{\lambda \in \Lambda_{f,1}} g(T +_{F_f} \lambda)$$

for every $g \in \mathcal{O}_K((T))$*. Moreover,* $\mathcal{N}_{F_f}$ *is continuous and multiplicative.*

In terms of Coleman norm operator, Ando's criterion becomes

$$[p]_{F_f}(T) = \mathcal{N}_{F_f}(T) \circ [p]_{F_f}(T)$$

$$\mathcal{N}_{F_f}(T) = T$$

Proof of Ando's Theorem in the Special Case

Want to show: given a Lubin-Tate formal group law F_f, there is a unique F_g in the $\star$-isomorphism class such that

$$\mathcal{N}_{F_g}(T) = T$$

If $u\colon F_f \to F_g$ is a $\star$-isomorphism, it can be shown that

$$\mathcal{N}_{F_g}(T) = T \Leftrightarrow \mathcal{N}_{F_f}\big(u(T)\big) = u(T)$$

We are left to show that there is a unique $u(T) \in T + T\pi[\![T]\!]$ fixed by $\mathcal{N}_{F_f}$.

Proof of Ando's Theorem in the Special Case

Want to show: $\exists\,!\ u(T) \in T + T\pi[\![T]\!]$ such that $\mathscr{N}_{F_f}\big(u(T)\big) = u(T)$.

A natural way to find fixed points is to take the limit.

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In fact, the operator $\mathscr{N}_{F_f}^{\infty}$ is well-defined on $\mathcal{O}_K(\!(T)\!)^*$.

And there is an exact sequence of groups:

$$1 \to 1 + \pi[\![T]\!] \to \mathcal{O}_K(\!(T)\!)^* \overset{\mathscr{N}_{F_f}^{\infty}}{\to} \left(\mathcal{O}_K(\!(T)\!)^*\right)^{\mathscr{N}_{F_f}} \to 1$$

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And there is an exact sequence of groups:

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Let $u(T) = \mathcal{N}_{F_f}^{\infty}(T)$ is the desired element.

Uniqueness: if $\mathcal{N}_{F_f}\big(u(T)\big) = u(T)$, $u(T) = \mathcal{N}_{F_f}^{\infty}\big(u(T)\big) = \mathcal{N}_{F_f}^{\infty}(T)$.

Proof of Ando's Theorem

Actually the above proof only depends on the properties of $\mathscr{N}_{F_f}$ and the fact that $[p]_{F_f}$ can be canceled from right.

We can generalize $\mathscr{N}_F$ to arbitrary complete local domain with $p \neq 0$, residue field containing k and F a lifting of arbitrary formal group law of finite height over k.

In particular, the Lubin-Tate ring $\mathscr{R}$ satisfies such condition.

Norm-Coherence in terms of Coleman Norm Operator

The motivation of Coleman norm operator is to prove an interpolation theorem on local fields, generating a function mapping division values v_n to a norm-coherent sequence.

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The motivation of Coleman norm operator is to prove an interpolation theorem on local fields, generating a function mapping division values v_n to a norm-coherent sequence.

Recall $\{\text{roots of } [p^n]_f\} =: \Lambda_{f,n} \cong \mathcal{O}_K/\pi^n$.

Let v_n be a generator of $\Lambda_{f,n}$ as a $\mathcal{O}_K$-module, s.t. $v_n = [p]_f(v_{n+1})$.

Then $\mathscr{N}_{F_f}(u) = u \Leftrightarrow N_{K_{\pi,n+1}/K_{\pi,n}}\big(u(v_{n+1})\big) = u(v_n)$ for all n.

In particular, $\mathscr{N}_{F_f}(T) = T \Leftrightarrow N_{K_{\pi,n+1}/K_{\pi,n}}(v_{n+1}) = v_n$ for all n.

Norm-Coherence in terms of Coleman Norm Operator

Example

Consider the case of $K = \mathbb{Q}_p$.

When $p > 2$, $F(X, Y) = X + Y + XY$ is norm-coherent, $v_n = \mu_{p^n} - 1$.

When $p = 2$, $F(X, Y) = X + Y - XY$ is norm-coherent, $v_n = \mu_{p^n} + 1$.